Linear Algebra

[KOMS119602] - 2022/2023

12.3 - Properties of Linear Transformation

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Learning objectives

After this lecture, you should be able to:

1. explain various properties of each of linear transformations in a vector space.

Properties of Matrix Transformations

(page 270 of Elementary LA Applications book)

Compositions of matrix transformation

Let:

- T_A: a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^k$
- T_B: a matrix transformation from $\mathbb{R}^k$ to $\mathbb{R}^m$

Let $\mathbf{x} \in \mathbb{R}^n$, and defines transformation:

$$\mathbf{x} \xrightarrow{T_A} T_A(\mathbf{x}) \xrightarrow{T_B} T_B(T_A(\mathbf{x}))$$

defines the transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

It is called the composition of T_B with T_A and is denoted by $T_B \circ T_A$. So:

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$$

Compositions of matrix transformation

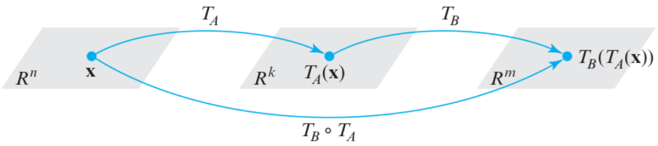
The composition is a matrix transformation, since:

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) = B(T_A(\mathbf{x})) = B(A\mathbf{x}) = (BA)\mathbf{x}$$

meaning that the result of the composition to $\mathbf{x}$ is obtained by multiplying $\mathbf{x}$ with BA on the left.

This is denoted by:

$$T_B \circ T_A = T_{BA}$$



Composition of three transformations

Compositions can be defined for any finite succession of matrix transformations whose domains and ranges have the appropriate dimensions. For instance, given:

$$T_A : \mathbb{R}^n \to \mathbb{R}^k, \;\; T_B : \mathbb{R}^k \to \mathbb{R}^\ell, T_C : \mathbb{R}^\ell \to \mathbb{R}^m$$

we can define the composition:

$$(T_C \circ T_B \circ T_A) : \mathbb{R}^n \to \mathbb{R}^m$$

by:

$$(T_C \circ T_B \circ T_A)(\mathbf{x}) = T_C(T_B(T_A(\mathbf{x})))$$

It can be shown that this is a matrix transformation with standard matrix CBA, and:

$$T_C \circ T_B \circ T_A = T_{CBA}$$

Notation

We can write the standard matrix for transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ without specifying the name of the standard matrix.

It is often written as $[T]$.

For instance,

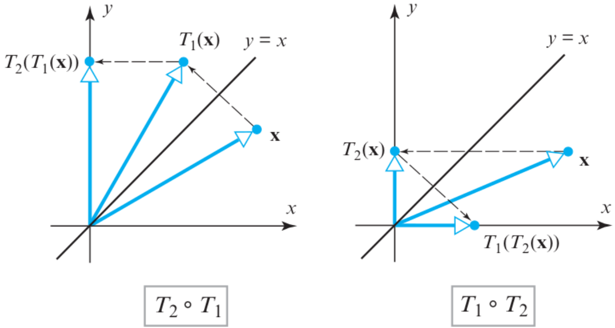
- $T(\mathbf{x}) = [T]\mathbf{x}$
- $[T_2 \circ T_1] = [T_2][T_1]$
- $[T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1]$

Composition is not commutative

Example

Let:

- $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y = x$;
- $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the y-axis.



Geometrically, both transformations have different effect on $\mathbf{x}$

Composition is not commutative (*cont.*)

Algebraically, we can compute:

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Clearly, $[T_1 \circ T_2] \neq [T_2 \circ T_1]$.

Composition of rotation is commutative

Example

Given :

$$T_1 : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{and} \quad T_2 : \mathbb{R}^2 \to \mathbb{R}^2$$

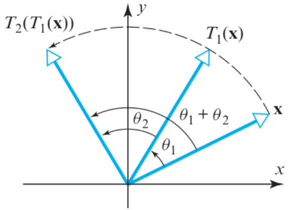
the matrix operators that rotate vectors about the origin through the angles θ_1 and θ_2 respectively.

So, the operation:

$$T_2 \circ T_1(\mathbf{x}) = T_2(T_1(\mathbf{x}))$$

first rotates x through the angle θ_1, then rotates $T_1(\mathbf{x})$ through the angle θ_2.

Hence, $(T_2 \circ T_1)(\mathbf{x})$ defines rotation of $\mathbf{x}$ through the angle $\theta_1 + \theta_2$.



Composition of rotation is commutative (*cont.*)

In this case, we have:

$$[T_1] = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \quad \text{and} \quad [T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

We show that: $[T_2 \circ T_1] = [T_2][T_1]$

$$\text{Note that} \quad [T_2 \circ T_1] = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

Furthermore:

$$\begin{aligned} [T_2][T_1] &= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -(\cos\theta_2\sin\theta_1 + \sin\theta_2\cos\theta_1) \\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & -\sin\theta_2 sin\theta_1 + \cos\theta_2\cos\theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \\ &= [T_2 \circ T_1] \end{aligned}$$

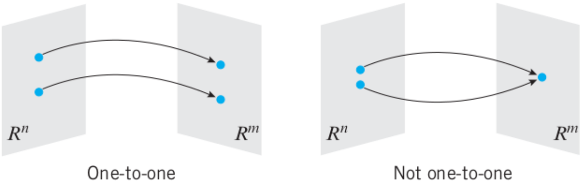
It can be easily seen that $[T_2 \circ T_1] = [T_1 \circ T_2]$ (hence, commutative).

Exercise

Read Example 3 and Example 4 (page 272-273)

One-to-one matrix transformation

A matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if T_A maps distinct vectors (points) in $\mathbb{R}^n$ into distinct vectors (points) in $\mathbb{R}^m$.



Equivalent statements:

- T_A is one-to-one if $\forall \mathbf{b}$ in the range of A, there is exactly one vector $\mathbf{x} \in \mathbb{R}^n$, s.t. $T_A\mathbf{x} = \mathbf{b}$.
- T_A is one-to-one if the equality $T_A(\mathbf{u}) = T_A(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.

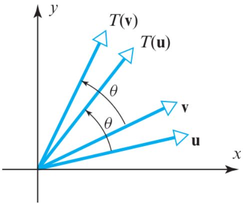
Examples: one-to-one and not one-to-one transformations

Rotation operators on $\mathbb{R}^2$ are one-to-one.

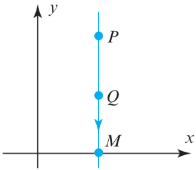
since distinct vectors that are rotated through the same angle have distinct images.

The orthogonal projection of $\mathbb{R}^2$ onto the x-axis is not one-to-one.

since it maps distinct points on the same vertical line into the same point.



▲ Figure 4.10.6 Distinct vectors $\mathbf{u}$ and $\mathbf{v}$ are rotated into distinct vectors $T(\mathbf{u})$ and $T(\mathbf{v})$.



▲ Figure 4.10.7 The distinct points P and Q are mapped into the same point M.

Kernel and range

If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, then the set of all vectors in RR^n that T_A maps into 0 is called the kernel of T_A and is denoted by $\ker(T_A)$, i.e.:

$$\ker(T_A) = \{\mathbf{x} \in \mathbb{R}^n \text{ s.t. } A\mathbf{x} = \mathbf{0}\}$$

The set of all vectors in $\mathbb{R}^m$ that are images under this transformation of at least one vector in $\mathbb{R}^n$ is called the range of T_A and is denoted by $R(T_A)$, i.e.:

$$R(T_A) = \{\mathbf{b} \in \mathbb{R}^m \text{ s.t. } \exists \mathbf{x} \in \mathbb{R}^n, \text{ where } A\mathbf{x} = \mathbf{b}\}$$

In brief:

$$\ker(T_A) = \text{null space of } A$$
$$R(T_A) = \text{column space of } A$$

Matrix - linear system - transformation

Let A be an $(m \times n)$ matrix.

Three ways of viewing the same subspace of $\mathbb{R}^n$:

- **Matrix view:** the null space of A
- **System view:** the solution space of $A\mathbf{x} = 0$
- **Transformation view:** the kernel of T_A

Three ways of viewing the same subspace of $\mathbb{R}^m$:

- **Matrix view:** the column space of A
- **System view:** all $\mathbf{b} \in \mathbb{R}^m$ for which $A\mathbf{x} = \mathbf{b}$ is consistent
- **Transformation view:** the range of T_A

Exercise

Read Example 5 and Example 6 on page 275.

One-to-one matrix operator

Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be a one-to-one matrix operator. So, A is invertible.

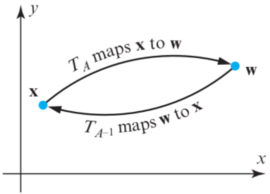
The inverse operator or the inverse of T_A is defined as:

$$T_{A^{-1}} : \mathbb{R}^n \to \mathbb{R}^n$$

In this case:

$$T_A(T_{A^{-1}}(\mathbf{x})) = AA^{-1}\mathbf{x} = I\mathbf{x} = \mathbf{x} \text{ or, equivalently } T_A \circ T_{A^{-1}} = T_{AA^{-1}} = T_I$$

$$T_{A^{-1}}(T_A(\mathbf{x})) = A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x} \text{ or, equivalently } T_{A^{-1}} \circ T_A = T_{A^{-1}A} = T_I$$



T_A maps $\mathbf{x}$ to $\mathbf{w}$ and $T_{A^{-1}}$ maps $\mathbf{w}$ back to $\mathbf{x}$, i.e., $T_{A^{-1}}(\mathbf{w}) = T_{A^{-1}}(T_A(\mathbf{x})) = \mathbf{x}$

Exercise

Read Example 7 and Example 8 on page 276.

Conclusion

THEOREM 4.10.2 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) A *is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row echelon form of* A *is* I_n.

(d) A *is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

(g) $\det(A) \neq 0$.

(h) *The column vectors of* A *are linearly independent.*

(i) *The row vectors of* A *are linearly independent.*

(j) *The column vectors of* A *span* R^n.

(k) *The row vectors of* A *span* R^n.

(l) *The column vectors of* A *form a basis for* R^n.

(m) *The row vectors of* A *form a basis for* R^n.

(n) A *has rank* n.

(o) A *has nullity* 0.

(p) *The orthogonal complement of the null space of* A *is* R^n.

(q) *The orthogonal complement of the row space of* A *is* $\{\mathbf{0}\}$.

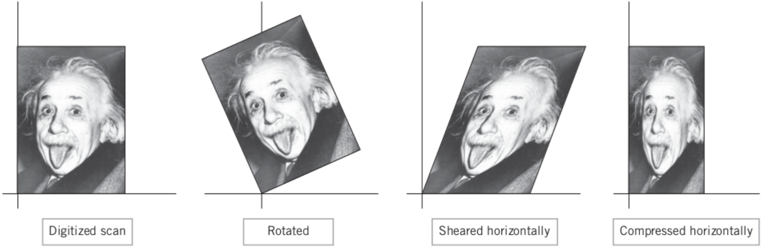
(r) *The kernel of* T_A *is* $\{\mathbf{0}\}$.

(s) *The range of* T_A *is* R^n.

(t) T_A *is one-to-one.*

Geometry of Matrix Operators on $\mathbb{R}^2$

(page 280 of Elementary LA Applications book) *to be continued...*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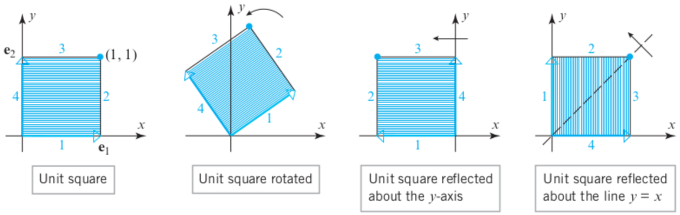
y
x
e_2
e_1
(1, 1)
1
2
3
4
Unit square
Unit square rotated
Unit square reflected about the y-axis
Unit square reflected about the line $y = x$

Exercise

Given a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is multiplication by an invertible matrix. Determine the image of:

1. A straight line
2. A line through the origin
3. Parallel lines
4. The line segment joining points P and Q
5. Three points lie on a line

Task:

Divide yourselves into 5 groups, and examine each of the question!

Exercises

Question 1

Given a transformation matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

Find the image of line $y = 2x + 1$ under the transformation.

Question 2

Given a transformation matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

Find the image of the unit square on the *first quadrant* under the transformation.

Exercises

Determine the image of the unit square under the following transformation:

- Reflection about the y-axis
- Reflection about the x-axis
- Reflection about the line $y = x$
- Rotation about the origin through a positive angle θ
- Compression in the x-direction with factor k with $0 < k < 1$
- Compression in the y-direction with factor k with $0 < k < 1$
- Expansion in the x-direction with factor k with $k > 1$
- Expansion in the y-direction with factor k with $k > 1$
- Shear in the x-direction with factor k with $k > 0$
- Shear in the x-direction with factor k with $k < 0$
- Shear in the y-direction with factor k with $k > 0$
- Shear in the y-direction with factor k with $k < 0$

This is the end of slide...